Math 3, Midterm Exam 1, Question 11

Question 11 a)

From the textbook, we know that polynomial, rational and trigonometric functions are continuous on their domain, which is the set of all real numbers.
We now analyze the branch for $x > 0$.

- $\frac{3\pi}{x}$ is a rational function, hence continuous on its domain, the set of all real numbers except 0.
- $\sin(x)$ is a trigonometric function, hence continuous on its domain, which is the set of all real numbers.
- Thus, $\sin(x) + \frac{3\pi}{x}$ is continuous on the set of all real numbers except 0, as it is a sum of two continuous functions.
- Moreover, $f(x) = \sin(x) + \frac{3\pi}{x}$ for $x > 0$, hence $f(x)$ is continuous for $x > 0$.

We now analyze the branch for $x \leq 0$.

- $2x$ is a linear function, hence continuous on its domain, which is the set of all real numbers.
- Thus, since $f(x) = 2x$ for $x \leq 0$, we have that $f(x)$ is continuous for $x < 0$ (note that we don't have continuity for free at 0).

So far, we have shown that $f(x)$ is continuous the set of all real numbers except 0. Now we need to address the continuity at $x = 0$, for that we recall the definition, $f(x)$ is continuous at 0 if

$$\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x).$$

We have

- $f(0) = 2 \cdot 0 = 0$.
- $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x = 2 \cdot 0 = 0$, as $f(x) = 2x$ for $x < 0$, and we know that $2x$ is a continuous function at 0.
- $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin(x) + \frac{3\pi}{x}$, and it does not exist, as when x gets closer to 0 from the right $3\pi/x$ is unbounded, indeed, it's limit is $+\infty$.
- Therefore, $f(x)$ is not continuous at 0.

Question 11 b)

We have

$$x \cdot f(x) = \begin{cases} x\left(\sin(x) + \frac{3\pi}{x}\right), & x > 0 \\ x \cdot (2x), & x \leq 0 \end{cases} = \begin{cases} x\sin(x) + 3\pi, & x > 0 \\ x \cdot 2x^2, & x \leq 0 \end{cases}$$

The limit exists if and only if, $\lim_{x\to 0^-} xf(x) = \lim_{x\to 0^+} xf(x)$.
However,

- $\lim_{x\to 0^-} xf(x) = \lim_{x\to 0^-} 2x^2 = 2 \cdot 0^2 = 0$.
- $\lim_{x\to 0^+} xf(x) = \lim_{x\to 0^+} x\sin(x) + 3\pi = 0 \cdot \sin(0) + 3\pi = 3\pi$.
- $3\pi \neq 0$, therefore, the limit does not exist

In both cases, we use the fact that $2x^2$ and $x\sin(x) + 3\pi$ are continuous function at $x = 0$ (do you know why?), and thus we can just evaluate them to compute their limit as $x \to 0$.

Question 11 c)

As above, we need to compute $\lim_{x\to 0^-} x^2 f(x)$ and $\lim_{x\to 0^+} x^2 f(x)$.
We have

$$x^2 \cdot f(x) = \begin{cases} x^2\left(\sin(x) + \frac{3\pi}{x}\right), & x > 0 \\ x^2 \cdot (2x), & x \leq 0 \end{cases} = \begin{cases} x^2 \sin(x) + 3\pi x, & x > 0 \\ x \cdot 2x^3, & x \leq 0 \end{cases}$$

and

- $\lim_{x\to 0^-} xf(x) = \lim_{x\to 0^-} 2x^3 = 2 \cdot 0^3 = 0$.
- $\lim_{x\to 0^+} xf(x) = \lim_{x\to 0^+} x^2 \sin(x) + 3\pi x = 0 \cdot \sin(0) + 3\pi \cdot 0 = 0$.
- Therefore, the limit exists, and its value is 0.

Again, we use the fact that $2x^3$ and $x^2 \sin(x) + 3\pi x$ are continuous function at $x = 0$ (do you know why?) to compute their limit as $x \to 0$.

12. (10 points) Consider the curve $x^2 = e^y - 2y$. In each of the following parts, remember to justify your answers completely.

(a) (2 points) Verify that $(1, 0)$ is on the curve.

$x = 1, \quad y = 0$

$\Rightarrow \quad x^2 = 1^2 = 1, \quad e^y - 2y = e^0 - 2\cdot 0$
$= 1 - 0 = 1$

$1 = 1 \quad \checkmark$

(b) (8 points) Find the slope of the tangent line to the curve at $(1, 0)$.

We find $\frac{dy}{dx}$ using implicit differentiation

$x^2 = e^y - 2y$

$$\Rightarrow \underbrace{\frac{d}{dx}(x^2)}_{2x} = \underbrace{\frac{d}{dx}(e^y)}_{e^y \cdot \frac{dy}{dx}} - \underbrace{\frac{d}{dx}(2y)}_{2\frac{dy}{dx}}$$

(by chain rule)

$$\Rightarrow \quad 2x = e^y \cdot \frac{dy}{dx} - 2\frac{dy}{dx}$$

Now we can solve for $\frac{dy}{dx}$.

$$2x = (e^y - 2)\cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x}{e^y - 2}.$$

To get the slope of the tangent line at $(1, 0)$, plug in $x = 1, y = 0$ into $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2\cdot 1}{e^0 - 2} = \frac{2}{-1} = \boxed{-2}.$$

13. (20 points) Consider the function $f(s) = 4s - 1$ near the point $s = 2$. In each of the following parts, remember to justify your answers completely.

   (a) (10 points) Let $\epsilon = 1$, $a = 2$, and $L = 7$. Find a $\delta > 0$ that satisfies the inequalities for this ϵ in the formal definition of the limit.

   Solution: To satisfy the inequalities in the definition of the limit for $\epsilon = 1$, we need to find a $\delta > 0$ so that when $2 - \delta < x < 2 + \delta$, we have that $6 < f(x) < 8$. We'll begin with the latter inequalities and use algebra to transform it into something as similar as possible to the first inequalities so that we find the range of x values for which the values of $f(x)$ are in the desired range:

   $$6 < 4x - 1 < 8$$

   Adding 1 to all parts of the inequalities yields,

   $$7 < 4x < 9,$$

   and dividing by 4 gives

   $$\frac{7}{4} < x < \frac{9}{4}$$

   So, for this range of x values, $f(x)$ is between the desired bounds of 6 and 8. To find a suitable δ, we need the interval $2 - \delta < x < 2 + \delta$ inside of this range or $\frac{7}{4} \leq 2 - \delta < x < 2 + \delta \leq \frac{9}{4}$. As we can rewrite $\frac{7}{4} = 2 - \frac{1}{4}$ and $\frac{9}{4} = 2 + \frac{1}{4}$, we can pick any $\delta \leq \frac{1}{4}$ and the inequalities will be satisfied.

   (b) (10 points) Use the formal definition of the limit to show that

   $$\lim_{s \to 2} 4s - 1 = 7$$

   .

Solution: To satisfy the definition of the limit, we need to show that for $\epsilon > 0$, we can find a $\delta > 0$ so that when $2 - \delta < x < 2 + \delta$, we have that $7 - \epsilon < f(x) < 7 + \epsilon$. We'll begin with the latter inequalities and use algebra to transform it into something as similar as possible to the first inequalities so that we find the range of x values for which the values of $f(x)$ are in the desired range:

$$7 - \epsilon < 4x - 1 < 7 + \epsilon$$

Adding 1 to all parts of the inequalities yields,

$$8 - \epsilon < 4x < 8 + \epsilon,$$

and dividing by 4 gives

$$2 - \frac{\epsilon}{4} < x < 2 + \frac{\epsilon}{4}$$

So, for this range of x values, $f(x)$ is between the desired bounds. To find a suitable δ, we need the interval $2-\delta < x < 2+\delta$ inside of this range or $2-\frac{\epsilon}{4} \leq 2-\delta < x < 2+\delta \leq 2+\frac{\epsilon}{4}$. So, we can pick any $\delta \leq \frac{\epsilon}{4}$ and the inequalities, and hence the definition, will be satisfied.

Math 3, Midterm Exam 1, Question 14

The definition of $g'(x)$ is the following:

$$g'(a) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \text{ or } \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$$

$$\begin{aligned}
g'(a) &= \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\
&= \lim_{h\to 0} \frac{((a+h)^2+1)-(a^2+1)}{h} && \text{(evaluating } g\text{)} \\
&= \lim_{h\to 0} \frac{(a^2+2ah+h^2+1)-(a^2+1)}{h} && \text{(expanding the square)} \\
&= \lim_{h\to 0} \frac{2ah+h^2}{h} && \text{(simplifying the numerator)} \\
&= \lim_{h\to 0} \frac{h(2a+h)}{h} && \text{(factoring out } h \text{ on the numerator)} \\
&= \lim_{h\to 0} \frac{2a+h}{1} && \text{(removing common factors)} \\
&= \lim_{h\to 0} 2a+h \\
&= 2a+0 && \text{(evaluating the limit of a linear function)} \\
&= 2a
\end{aligned}$$

Math 3, Midterm Exam 1, Question 15

Question 15 a)

From the textbook, we know that polynomial and trigonometric functions are continuous on their domain, which is the set of all real numbers.

$\frac{\pi x}{2}$, $x+3$, x^2+9 are polynomial functions, and $\cos(x)$ is a trigonometric function. Hence they are continuous on their domain, which is the set of all real numbers. $\cos(x)$ is a trigonometric function, hence continuous on its domain, which is the set of all real numbers.

$\cos(\frac{\pi x}{2})$ is the composition of $\cos x$ and $\frac{\pi x}{2}$, both continuous on their domain, therefore $\cos(\frac{\pi x}{2})$ is continuous on its domain, the set of all real numbers.

$h(x)$ is not defined when the denominator is equal to 0. This happens when $(x^2-9)=0$. Hence, the domain of $h(x)$ is the set of all real numbers, except for $x=-3,3$.

Finally, from the textbook, we know that the quotient of two continuous functions is continuous on its domain. So, $h(x)$ is continuous on the set of all real numbers except for $x=-3,3$.

Question 15 b)

When considering the set of all real numbers, there are two points at which $h(x)$ is discontinuous, $x=3$ and $x=-3$.

$x=-3$ is a removable discontinuity because we can redefine $h(x)=\frac{(x+3)cos(\frac{\pi x}{2})}{(x^2-9)}=\frac{(x+3)cos(\frac{\pi x}{2})}{(x-3)(x+3)}=\frac{cos(\frac{\pi x}{2})}{(x-3)}$ to remove the discontinuity. Alternatively, we can also "plug in the hole" by defining $h(-3)=\frac{cos(\frac{\pi x}{2})}{(x-3)}=\frac{0}{-6}=0$.

$x=3$ is also a removable discontinuity. This is because if we set $f(x)=\frac{cos(\frac{\pi x}{2})}{(x-3)}=\frac{cos(\frac{\pi x}{2})-cos(\frac{\pi 3}{2})}{(x-3)}$, then $\lim_{x\to 3}f(x)=cos'(\frac{\pi x}{2})|_{x=3}=-\frac{\pi}{2}sin(\frac{\pi 3}{2})=\frac{\pi}{2}$. (the trick here is to recognize that $f(x)$ is the derivative of cosine at a point) So, the limit exists and we can remove the discontinuity by setting $h(3)=\frac{\pi}{2}$.

Question 15 c)

We can either use the original function $h(x)$ or use the redefined function (much easier). Let $g(x)=\frac{cos(\frac{\pi x}{2})}{(x-3)}$ be the redefined function.

Then $g'(x)=\frac{1}{(x-3)^2}\cdot(-sin(\frac{\pi x}{2})\frac{\pi}{2}(x-3)-cos(\frac{\pi x}{2}))$ by the quotient rule and the chain rule.

Finally, plug in $x=2$ into the derivative to get $g'(2)=\frac{1}{(2-3)^2}\cdot(-sin(\frac{\pi 2}{2})\frac{\pi}{2}(2-3)-cos(\frac{\pi 2}{2}))=1\cdot(0-(-1))=1$.

$1=g'(2)=h'(2)$ since we know from algebraic simplification that $g=h$ everywhere except at the point $x=-3$, and hence must have the same derivative at the point $x=2$.