Math 3 Fall 2016

Midterm 2

October 27, 2016

Your name (please print): ____

Circle your section: Pauls (9L) Costa (10) Costa (11) Khoo (12) Troyka (2)

Instructions: This is a closed book, closed notes exam. **Use of calculators is not permitted**. Except for the multiple choice questions, you must justify all of your answers completely to receive credit - write clearly in complete sentences. You may not give or receive any help on this exam and all questions should be directed to your instructor.

You have **2 hours** to work on all **14** problems. Please do all your work in this exam booklet.

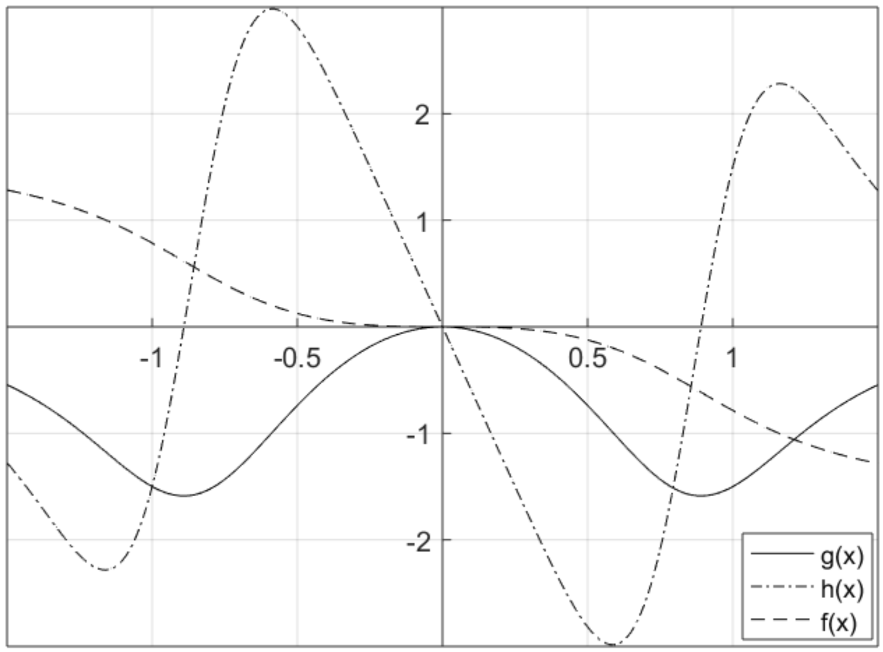
The Honor Principle requires that you neither give nor receive any aid on this exam.

Problem	Points	Score
1-10	30	
11	20	
12	20	
13	15	
14	15	
Total	**100**	

Answer Sheet for Multiple Choice Questions

Problem	Answer
1	A
2	A
3	B
4	C
5	C
6	A
7	A
8	C
9	A
10	D

1. Consider the graph of $f(x)$, $g(x)$, and $h(x)$ in the figure:



How are f, g, and h related?

A. $g(x) = f'(x)$ and $h(x) = g'(x) = f''(x)$.
B. $g(x) = h'(x)$ and $f(x) = g'(x) = h''(x)$.
C. $f(x) = g'(x)$ and $h(x) = f'(x) = g''(x)$.
D. $f(x) = h'(x)$ and $g(x) = f'(x) = h''(x)$.
E. $h(x) = f'(x)$ and $g(x) = h'(x) = f''(x)$.
F. $h(x) = g'(x)$ and $f(x) = h'(x) = g''(x)$.

2. Let $f(x) = |x|$. Does f have a critical point at $x = 0$?

A. Yes, because $f'(0)$ is undefined.
B. Yes, because $f'(0) = 0$.
C. No, because $f'(0) \neq 0$.
D. No, because $f'(0)$ is undefined.
E. No, because the graph of f has a corner at $x = 0$.

3. For which value of x does $f(x) = x \cdot \ln(x)$ reach its absolute minimum?

   A. 0

   B. $\frac{1}{e}$

   C. e

   D. 1

   E. $f(x)$ does not have an absolute minimum.

4. Let

$$f(x) = \frac{\sqrt{4x^2 - x}}{2x + 1}.$$

   Choose the correct statement.

   A. $f(x)$ has no horizontal asymptotes.

   B. $f(x)$ has a single horizontal asymptote at $y = 1$.

   C. $f(x)$ has two horizontal asymptotes, at $y = 1$ and at $y = -1$.

   D. $f(x)$ has a single horizontal asymptote at $y = -1$.

5. Which of the following statements is true? (**Hint:** Extreme Value Theorem)

   A. If $f(x)$ is a continuous function on a closed interval $[a, b]$, then the absolute maximum and absolute minimum occur at points c where $f'(c) = 0$.

   B. If $f(x)$ is a continuous function on an open interval (a, b), then $f(x)$ does not attain an absolute maximum nor an absolute minimum on the interval (a, b).

   C. If $f(x)$ is a continuous function on a closed interval $[a, b]$, then $f(x)$ attains an absolute maximum and absolute minimum at some points in the interval $[a, b]$.

   D. Given a function $f(x)$ on a closed interval $[a, b]$, if $f(x)$ attains an absolute maximum and absolute minimum then $f(x)$ is a continuous function.

6. Let $f(x) = e^{x^2}$. Which of the following is true?

   A. f is concave up everywhere.

   B. f is concave down everywhere.

   C. f is concave up **only** on the interval $[0, \infty)$ and nowhere else.

   D. f has an inflection point at $x = 0$.

   E. None of the above.

7. Let
$$g(x) = e^{\frac{1}{1+x^2}}.$$
Does g have an absolute maximum value?

   A. Yes, the absolute maximum value is e.
   B. Yes, the absolute maximum value is 1.
   C. Yes, the absolute maximum value is 0.
   D. No, because $g(x)$ is unbounded when $x \to \pm\infty$.
   E. No, because $g(x)$ has a vertical asymptote.
   F. No, because the domain of g is not a closed interval so the Extreme Value Theorem does not apply.

8. Recall that the volume V of a sphere is $V = \frac{4}{3}\pi r^3$, where r is its radius. Air is being pumped into a spherical balloon at the rate of 2 cm^3 / min. What is the rate at which the radius of the balloon is increasing when the diameter is 10 cm?

   A. $\left(\frac{2}{\frac{4}{3}\pi}\right)^{\frac{1}{3}}$ cm / min
   B. $\frac{1}{25\pi}$ cm / min
   C. $\frac{1}{50\pi}$ cm / min
   D. $\frac{1}{10\pi}$ cm / min
   E. $\frac{1}{200\pi}$ cm / min

9. Find the maximum value of $x^3 - 12x$ on the interval $[-1, 1]$.

   A. 11
   B. 16
   C. 0
   D. 2
   E. -2
   F. Does not have a maximum.

10. What does it mean to say that a function f has an inflection point at $x = a$?

   A. $f''(a) = 0$ or $f''(a)$ is undefined.
   B. $f'(a) = 0$ or $f'(a)$ is undefined.
   C. $f''(a) = 0$
   D. At $x = a$, the concavity of f changes from up to down, or from down to up.
   E. At $x = a$, f changes from increasing to decreasing, or from decreasing to increasing.

Long answer questions

For each long answer problem, you must give a full explanation and justification of your answer to receive credit. A list of computations is not sufficient to gain credit!

11. (20 points)

$$f(x) = \begin{cases} 10 + 2x - x^2 & x \geq 0 \\ \frac{1}{x} & x < 0 \end{cases}$$

For parts a-d, justify your answers completely.

a) Find all the asymptotes of $f(x)$.
Solution: Vertical asymptotes may occur at points where the function is not defined or is discontinuous. For $x \geq 0$ the function is defined by a polynomial and as polynomials are defined and continuous for any x, the domain includes all $x \geq 0$. For $x < 0$, $\frac{1}{x}$ is defined as well and so the only place where we may have an issue is at $x = 0$, where the two parts of the function definition meet. We compute the left and right hand limits to explore the behavior of the function near zero. Since $10 + 2x - x^2$ is a polynomial and all polynomials are continuous where they are defined, we have that $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 10 - 2x - x^2 = 10$. From the other side, we have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x}$. As, when we plug in small negative numbers to $1/x$ we get large negative numbers, this limit tends to $-\infty$. So, we have a vertical asymptote at $x = 0$.

To check for horizontal asymptotes we compute two limits. First,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 10 + 2x - x^2 = \lim_{x\to\infty} x^2(10/x^2 + 2/x - 1).$$

Since the interior of the parenthesis tends to -1 and x tends to infinity, we conclude that $\lim_{x\to\infty} f(x) = -\infty$. Second,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x} = 0.$$

So, there is a horizontal asymptote at $y = 0$ as x tends to minus infinity.

b) Find and classify the critical points of $f(x)$, if any. On what interval(s) is f increasing? On what interval(s) is f decreasing?
We first calculate the derivative to find the critical numbers, looking for places where it is zero or does not exist. For $x > 0$, $f(x) = 10 + 2x - x^2$ so by the power rule we have $f'(x) = 2 - 2x$ for $x > 0$. When $x < 0$, $f(x) = 1/x$ and $f'(x) = -x^{-2}$. At $x = 0$, as the function is not continuous, the derivative does not exist. In part (a), we saw that the function has a vertical asymptote at $x = 0$, the critical point $(0, 10)$ is neither a maximum nor a minimum. Putting the two parts together, we see that the derivative is only zero at $x = 1$. As $f(1) = 11$, the critical point is $(1, 11)$.

To classify $(1, 11)$, we can use the first derivative test. To the left of $x = 1$, we test $x = 1/2$ and $f'(1/2) = 2 - 1 = 1 > 0$ so the function is increasing when approaching the critical point. To the right of $x = 1$, we test $x = 2$ and $f'(2) = 2 - 4 < 0$, so the function is decreasing after we pass $x = 1$. We conclude this critical point is a local maximum.

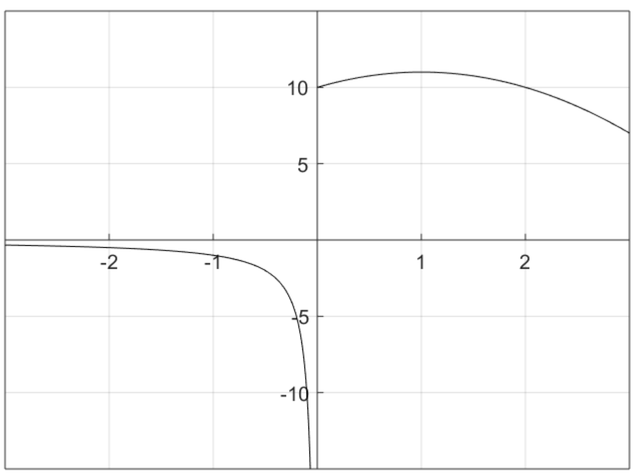
To finish out this part, we test a point to the left of $x = 0$ - as f' is not defined there, the sign may change when passing through. At $x = -1$, $f'(-1) = -1/1 = -1 > 0$ so f is decreasing to the left of $x = 0$.

Putting this together, we have that f is decreasing on $(-\infty, 0) \cup (1, \infty)$ and increasing on $(0, 1)$.

c) On what interval(s) is f concave up? On what interval(s) is f concave down?
To determine the concavity, we look at the sign of the second derivative. For $x > 0$, as $f'(x) = 2 - 2x$, $f''(x) = -2$ by the power rule. This is always negative so f is concave down on $(0, \infty)$. For $x < 0$, as $f'(x) = -x^{-2}$ we have $f''(x) = 2x^{-3}$ again by the power rule. As this function is negative for all negative values of x, we have f is also concave down on $(-\infty, 0)$. As f' is not defined at $x = 0$, neither is f''.

In summary, f is concave down on $(-\infty, 0) \cup (0, \infty)$.

d) Sketch the curve of $f(x)$ using all the information from the previous parts.



12. (20 points) A school is building a track that encloses a rectangular field with a semicircle on each end, as shown in the diagram. The track team requires the track be 400 meters in length. What is the largest possible area of the field (the shaded rectangular region in the figure)?

Solution: If we let the radius of the semi-circles be r, then the short side of the rectangle has length $2r$. If we further let the long side of the rectangle have length w, then the perimeter of the region (and hence the length of the track) is $2w + 2\pi r = 400$. The area of the shaded region is $A = 2rw$. We wish to maximize the area given the constraint on the perimeter, so we solve the perimeter equation for w yielding $w = (400 - 2\pi r)/2 = 200 - \pi r$. Plugging this into the area equation gives $A(r) = 2r(200 - \pi r) = 400r - 2\pi r^2$. Given that r is a physical parameter describing a length, the smallest it can be is zero. The largest it can be is achieved when w is zero and the two semi-circles take up the entire perimeter - when $2\pi r = 400$ or $r = 200/\pi$. So, we are left with the problem of finding the global maximum of $A(r) = 400r - 2\pi r^2$ when $0 \leq r \leq 200/\pi$.

To solve this problem, we use the process associated to the Extreme Value Theorem -we find all critical points in the specified domain, plug the critical points and endpoints into A, and select the maximum value. Critical points occur when the derivative is zero or undefined. The derivative is $A'(r) = 400 - 4\pi r$, which equals zero when $r = 400/4\pi = 100/\pi$, and it is never undefined. The critical number is inside the specified domain. As $A(0) = 0$, $A(200/\pi) = (400 \cdot 200 - 2 \cdot 200^2)/\pi = 0$, and $A(100/\pi) = (400 \cdot 100 - 2 \cdot 200^2)/\pi = 20000/\pi$, the maximum occurs when $r = 100/\pi$ with an area of $20000/\pi$.

13. (15 points) The **surface area** of a cube is increasing at a rate of 10 m^2/s. At a particular moment, the surface area is 54 m^2. What is the rate of change of the **volume** of the cube at the same moment?

Solution: If the side length of the cube is s, then the volume is $V = s^3$ and the surface area is $S = 6a^2$. We'd like the volume written in terms of the surface area so we solve the equation for the latter for s, yielding $s = \sqrt{S/6}$. Plugging this into the volume equation gives $V = \left(\frac{S}{6}\right)^{\frac{3}{2}}$.

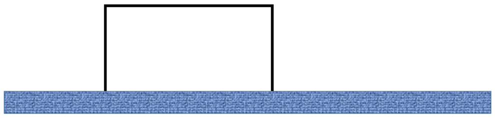
Differentiating V with respect to time, we have that

$$\frac{dV}{dt} = \frac{3}{2}\left(\frac{S}{6}\right)^{\frac{1}{2}} \cdot \frac{1}{6}\frac{dS}{dt}.$$

Plugging in the given information that at the specified moment in time, $S = 54$ and $dS/dt = 10$, we have

$$\frac{dV}{dt} = \frac{3}{2}\left(\frac{54}{6}\right)^{\frac{1}{2}} \frac{10}{6} = \frac{5}{2}\sqrt{9} = \frac{15}{2}.$$

14. (15 points) Given 20 feet of fencing, we want to fence off a rectangular area, so that one side of the rectangular borders a straight river and hence requires no fencing (see figure). What are the dimensions of the rectangle that maximizes the area?



Solution: Label the length of the top side of the rectangle x and the other side lengths y. Then, the area of the rectangle is $A = xy$ and the length of the fence is $2x + y = 20$ or $y = 20 - 2x$. Plugging this into the area equation yields $A = x(20 - 2x)$. Given the physical constraints that neither of the side lengths can be less than zero, we have that $0 \leq x \leq 10$. To find the absolute maximum, we use the process associated to the Extreme Value Theorem - we find all critical points in the specified domain, plug the critical points and endpoints into A, and select the maximum value. To find critical points, we take the derivative using the product rule: $A' = 20 - 2x - 2x = 20 - 4x$. This equals zero at $x = 5$, which is between zero and ten. Since $A(0) = 0 = A(10)$ and $A(5) = 5(20 - 10) = 50$, the maximum occurs when $x = 5$. By the formula above, when $x = 5$, we have $y = 10$, which together give the dimensions of the rectangle with the largest area given the constraints.

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